and

$$tg\left(a_{k+1}+a_k\right)=\frac{tga_{k+1}+tga_k}{1-tga_{k+1}tga_k}=\frac{P_{k+2}^2-P_k^2}{2\left(P_kP_{k+1}^2P_{k+2}-1\right)}$$

$$S_n=\sum_{k=1}^{n}arctg\frac{P_{k+2}^2-P_k^2}{2\left(P_kP_{k+1}^2P_{k+2}-1\right)}arctg\frac{2P_{k+1}^2}{P_kP_{k+1}^2P_{k+2}+1}=$$

$$=\sum_{k=1}^{n}arctg\left(tg\left(a_{k+1}-a_k\right)\right)arctg\left(tg\left(a_{k+1}+a_k\right)\right)=$$

$$=-\sum_{k=1}^{n}\left(a_{k+1}^2-a_k^2\right)=a_1^2-a_{n+1}^2$$

$\lim\limits_{n\to\infty}a_{n+1}^2=0$, therefore

$$\lim_{n\to\infty}S_n=a_1^2=\left(arctg\frac{1}{2}\right)^2$$

Mihály Bencze

Second solution. Let $\alpha_k:=\arctan\dfrac{1}{P_kP_{k+1}}$ and

$$S_n=\sum_{k=1}^{n}\arctan\frac{P_{k+2}^2-P_k^2}{2\left(P_kP_{k+1}^2P_{k+2}-1\right)}\cdot\arctan\frac{2P_{k+1}^2}{P_kP_{k+1}^2P_{k+2}+1}.$$

Then

$$\frac{P_{k+2}^2-P_k^2}{2\left(P_kP_{k+1}^2P_{k+2}-1\right)}=\frac{\left(P_{k+2}-P_k\right)\left(P_{k+2}+P_k\right)}{2\left(P_kP_{k+1}^2P_{k+2}-1\right)}=$$

$$=\frac{2P_{k+1}\left(P_{k+2}+P_k\right)}{2\left(P_kP_{k+1}^2P_{k+2}-1\right)}=\frac{\dfrac{1}{P_kP_{k+1}}+\dfrac{1}{P_{k+1}P_{k+2}}}{1-\dfrac{1}{P_kP_{k+1}}\cdot\dfrac{1}{P_{k+1}P_{k+2}}}=$$

$$=\frac{\tan\alpha_k+\tan\alpha_{k+1}}{1-\tan\alpha_k\cdot\tan\alpha_{k+1}}=\tan\left(\alpha_k+\alpha_{k+1}\right)$$

and

$$\frac{2P_{k+1}^2}{P_kP_{k+1}^2P_{k+2}+1} = \frac{P_{k+1}\left(P_{k+2}-P_k\right)}{P_kP_{k+1}^2P_{k+2}+1} = \frac{\dfrac{1}{P_kP_{k+1}} - \dfrac{1}{P_{k+1}P_{k+2}}}{1+\dfrac{1}{P_kP_{k+1}}\cdot\dfrac{1}{P_{k+1}P_{k+2}}} =$$

$$= \tan\left(\alpha_k - \alpha_{k+1}\right).$$

Since $P_n > 1$ for any $n \geq 2$ and P_n increase in $n \in \mathbb{N}$ then
$\alpha_k = \arctan\dfrac{1}{P_kP_{k+1}} \in (0, \pi/4)$
and $\alpha_{k+1} > \alpha_k$ for any $k \in \mathbb{N}$.Hence, $\alpha_k - \alpha_{k+1}, \alpha_k + \alpha_{k+1} \in (0, \pi/2)$ and then

$$\arctan\left(\tan\left(\alpha_k + \alpha_{k+1}\right)\right) = \alpha_k + \alpha_{k+1}, \arctan\left(\tan\left(\alpha_k - \alpha_{k+1}\right)\right) = \alpha_k - \alpha_{k+1}.$$

Thus,

$$\sum_{k=1}^{n} \arctan\frac{P_{k+2}^2 - P_k^2}{2\left(P_kP_{k+1}^2P_{k+2}-1\right)} \cdot \arctan\frac{2P_{k+1}^2}{P_kP_{k+1}^2P_{k+2}+1} =$$

$$= \sum_{k=1}^{n}\left(\alpha_k^2 - \alpha_{k+1}^2\right) = \alpha_1^2 - \alpha_{n+1}^2.$$

Since $\lim\limits_{n\to\infty} P_n = \infty$ ($P_{n+1} = 2P_n + P_{n-1}, n \in \mathbb{N} \implies P_{n+1} \geq 2P_n, n \in \mathbb{N} \implies P_n \geq 2^{n-1}, n \in \mathbb{N}$)
then $\lim\limits_{n\to\infty} \alpha_{n+1} = \lim\limits_{n\to\infty} \arctan\dfrac{1}{P_{n+1}P_{n+2}} = 0$ and, therefore,

$$\sum_{k=1}^{\infty} \arctan\frac{P_{k+2}^2 - P_k^2}{2\left(P_kP_{k+1}^2P_{k+2}-1\right)} \cdot \arctan\frac{2P_{k+1}^2}{P_kP_{k+1}^2P_{k+2}+1} = \lim_{n\to\infty} S_n =$$

$$= \alpha_1^2 - \lim_{n\to\infty}\alpha_{n+1}^2 = \alpha_1^2 = \left(\arctan\frac{1}{1\cdot 2}\right)^2 = \left(\arctan\frac{1}{2}\right)^2 = 0.214\,97.$$

Arkady Alt

Third solution. Since

$$P_{k+1}(P_{k+2} - P_k) = P_{k+1}(2P_{k+1} + P_k - P_k) = 2P_{k+1}^2$$